CONCURSUL NAȚIONAL DE MATEMATICĂ
„TEHNICI MATEMATICE”-ediția a XIX-a
Etapa națională 23.02.2024
Clasa a XII -a Matematică *M_șt-nat*
Barem de corectare

Subiectul I

1. $b_1b_8 + b_3b_6 = 256\sqrt{2} \Rightarrow 2b_1b_8 = 256\sqrt{2}$ 2p

$\Rightarrow b_1^2q^7 = 128\sqrt{2} \Rightarrow q^7 = 2^7 \Rightarrow q = 2$ 3p

2. $x_1 + x_2 = 2\left(4-\sqrt{3}\right), x_1 \cdot x_2 = 3$ 2p

$E^2 = x_1 + x_2 + 2\sqrt{x_1x_2} = 8 \Rightarrow E = 2\sqrt{2}.$ 3p

3. $f(2)\cdot f(3)\cdot f(4)\cdot\ldots\cdot f(x) = \frac{\lg 3}{\lg 2}\cdot\frac{\lg 4}{\lg 3}\cdot\frac{\lg 5}{\lg 4}\cdot\ldots\cdot\frac{\lg(x+1)}{\lg x} = \frac{\lg(x+1)}{\lg 2} = \log_2(x+1)$ 3p

$2^{\log_2(x+1)} = 2024 \Rightarrow x = 2023$ 2p

4 $z_3 = a + bi, 1 = |z_1 + z_3| \Rightarrow (a+1)^2 + (b+1)^2 = 1, 1 = |z_2 + z_3| \Rightarrow (a+1)^2 + (b-1)^2 = 1$ 3p

$(b+1)^2 = (b-1)^2 \Rightarrow b = 0; a = -1 \Rightarrow z_3 = -1$ 2p

5. $m = \frac{3n}{n+1}$ 2p

$m = 3 - \frac{3}{n+1} \Rightarrow n+1 \in \{1,3\} \Rightarrow n \in \{0,2\}$, se obțin două drepte....................3p

6. $S = \frac{ab\sin C}{2} = \frac{ac\sin B}{2} = \frac{bc\sin A}{2}$ 2p

$6S = |6S| = |ab\sin C + ac\sin B + bc\sin A| \le ab + ac + bc$ 3p

Subiectul II

1.a) $\det(A(\alpha)) = 1 \ne 0 \Rightarrow A(\alpha)$ inversabilă....................2p

$A^{-1}(\alpha) = \begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$ 3p

b) Prin calcul $A(\alpha)A(\beta) = A\left(\alpha + \beta - \frac{\pi}{2}\right), \forall \alpha, \beta \in \mathbb{R}.$ 3p

$A\left(\frac{\pi}{4}\right)A\left(\frac{\pi}{6}\right) = A\left(\frac{\pi}{4} + \frac{\pi}{6} - \frac{\pi}{2}\right) = A\left(-\frac{\pi}{12}\right)$ 2p

c) $A^2(x) = A\left(2x - \frac{\pi}{2}\right), A^3(x) = A\left(3x - 2\frac{\pi}{2}\right)$ 2p

$A^n(x) = A\left(nx - (n-1)\frac{\pi}{2}\right), A^{n+1}(x) = A^n(x)A(x) = A\left(nx - (n-1)\frac{\pi}{2}\right)A(x) =$

$= A\left((n+1)x - n\frac{\pi}{2}\right).$ 3p

2.a) Din $f(e) = 1$ 3p

$\Rightarrow \frac{1+e}{1-e} = 1 \Rightarrow e = 0$ element neutru 2p

b) $x * (x+1) = 0 \Rightarrow f(x*(x+1)) = f(0)$ 2p

$\Rightarrow f(x)\cdot f(1+x)=1 \Rightarrow \dfrac{1+x}{1-x}\cdot\dfrac{2+x}{-x}=1 \Rightarrow x=-\dfrac{1}{2}$ 3p

c) Inversa funcției f este $f^{-1}:\mathbb{R}_+^* \to G,\ f^{-1}(x)=\dfrac{x-1}{x+1}$ 1p

Atunci pentru $x,y\in G, x*y=f^{-1}\left(f(x)f(y)\right)=\dfrac{f(x)f(y)-1}{f(x)f(y)+1}=$ 2p

$=\dfrac{1+y+x+xy-(1-y-x+xy)}{1+y+x+xy+1-y-x-xy}=\dfrac{x+y}{1+xy}$. 2p

Subiectul III

1.a) $f'(x)=\dfrac{(\ln x)'(1+\ln x)-(\ln x)(1+\ln x)'}{(1+\ln x)^2}=$ 2p

$=\dfrac{\frac{1}{x}(1+\ln x)-\ln x\frac{1}{x}}{(1+\ln x)^2}=\dfrac{1}{x(1+\ln x)^2}$ 3p

b) $f''(x)=-\dfrac{3+\ln x}{x^2(1+\ln x)^3}$ 2p

$x>\dfrac{1}{e}\Rightarrow \ln x>-1\Rightarrow 1+\ln x>0, 3+\ln x>0;\ f''(x)\le 0\Rightarrow f$ concavă....................3p

c) Ecuația tangentei la grafic în punctul de abscisă a este $y-f(a)=f'(a)(x-a)$ 1p

Tangenta trece prin origine $.\,0-f(a)=f'(a)(-a)\Rightarrow f'(a)=\dfrac{f(a)}{a}\Rightarrow \ln^2 a+\ln a-1=0$...3p

Abscisa punctului este $a=e^{\frac{-1+\sqrt{5}}{2}}\in\left(\dfrac{1}{e},\infty\right)$. 1p

2.a) $\displaystyle\int_0^{\frac{\pi}{2}}\sin 2x f(\sin x)dx=\int_0^{\frac{\pi}{2}}\sin 2x e^{\sin^2 x}dx=$ 2p

Notăm $\sin^2 x=t$ și avem $\displaystyle\int_0^1 e^t dt=e-1$ 3p

b) Fie F o primitivă a funcției $f, g'(t)=\left(F\left(\sqrt{\ln t}\right)-F(0)\right)'=F'\left(\sqrt{\ln t}\right)\left(\sqrt{\ln t}\right)'=$ 3p

$=f\left(\sqrt{\ln t}\right)\dfrac{1}{2\sqrt{\ln t}}\dfrac{1}{t}=\dfrac{1}{2\sqrt{\ln t}}>0, \forall t\in(1,\infty)\Rightarrow g$ strict crescătoare pe $(1,\infty)$ 2p

c) $\displaystyle\int_0^1 f(x)dx=\int_0^1 e^{x^2}dx\le\int_0^1 e^x dx=e-1$ 2p

$e^x\ge x+1, \forall x\ge 0\Rightarrow e^{x^2}\ge x^2+1, \displaystyle\int_0^1 f(x)dx\ge\int_0^1\left(x^2+1\right)dx=\dfrac{4}{3}$ 3p