Colegiul Naţional „Mircea cel Bătrân", Râmnicu-Vâlcea
Concursul Interjudețean „Mathematica – Modus Vivendi"
Ediţia a XIX-a, 23 martie 2024
**BAREM CLASA a XII -a**

**1.a.** Facem schimbarea de variabilă $\frac{4}{x}=t \Rightarrow x=\frac{4}{t} \Rightarrow dx=-\frac{4}{t^2}dt$

$I=\int_4^1 \frac{\ln\left(\frac{4}{t}\right)}{\frac{16}{t^2}+4}\left(-\frac{4}{t^2}\right)dt=\int_1^4 \frac{\ln 4-\ln t}{4t^2+16}\cdot t^2\cdot\left(\frac{4}{t^2}\right)dt=\int_1^4 \frac{\ln 4-\ln t}{t^2+4}dt$.........1p

$I=\ln 4\cdot\int_1^4 \frac{1}{t^2+4}dt-\int_1^4 \frac{\ln t}{t^2+4}dt$.........1p

$I=\ln 4\cdot\frac{1}{2}arctg\frac{t}{2}\Big|_1^4-I \Rightarrow 2I=\frac{2\ln 2}{2}\cdot\left(arctg\,2-arctg\frac{1}{2}\right)$

$I=\frac{\ln 2}{2}\cdot(arctg\,2-arctg\frac{1}{2})$.........1p

**b.** Facem schimbarea de variabilă $x=\frac{1}{t}$ şi avem

$I=\int_{\frac{1}{a}}^{a}\frac{arctg\frac{1}{t}}{t^2+3t+1}dt=\int_{\frac{1}{a}}^{a}\frac{\frac{\pi}{2}-arctg\,x}{x^2+3x+1}dx=\frac{\pi}{2}\int_{\frac{1}{a}}^{a}\frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}-I$ , de unde.........1p

$I=\frac{\pi}{4}\cdot\frac{1}{\sqrt{5}}\ln\left|\frac{x+\frac{3}{2}-\frac{\sqrt{5}}{2}}{x+\frac{3}{2}+\frac{\sqrt{5}}{2}}\right|\Bigg|_{\frac{1}{a}}^{a}=\frac{\pi}{4\sqrt{5}}\cdot\left(\ln\left|\frac{2a+3-\sqrt{5}}{2a+3+\sqrt{5}}\right|-\ln\left|\frac{\frac{2}{a}+3-\sqrt{5}}{\frac{2}{a}+3+\sqrt{5}}\right|\right)=$.........2p

$=\frac{\pi}{4\sqrt{5}}\cdot ln\frac{(2a+3-\sqrt{5})\left(2+a(3+\sqrt{5})\right)}{(2a+3+\sqrt{5})\left(2+a(3-\sqrt{5})\right)}$.........1p

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Total = 7 puncte

**2. a.** $\int\frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x^2+2x+2025+\frac{2}{x}+\frac{1}{x^2}\right)}dx=\int\frac{\left(1-\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)+2025}dx$.........1p

Facem schimbarea de variabilă $x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx=dt \Rightarrow x^2+\frac{1}{x^2}=t^2-2$.........1p

$I=\int\frac{dt}{t^2-2+2t+2025}=\int\frac{dt}{t^2+2t+2023}=\int\frac{dt}{(t+1)^2+2022}=\frac{1}{\sqrt{2022}}arctg\frac{t}{\sqrt{2022}}$

$I=\frac{1}{\sqrt{2022}}arctg\frac{x+\frac{1}{x}}{\sqrt{2022}}+C$.........1p

**b.** Avem $\left(\frac{a^x\cdot\sin bx}{a^x+\sin bx}\right)'=\frac{1}{(a^x+\sin bx)^2}\cdot[(a^x\cdot\ln a\cdot\sin bx+a^x\cdot b\cdot\cos bx)(a^x+\sin bx)-a^x\cdot\sin bx\cdot(a^x\cdot\ln a+b\cdot\cos bx)]=$.........1p

$=\frac{a^x}{(a^x+\sin bx)^2}(a^x\cdot\ln a\cdot\sin bx+b\cdot a^x\cdot\cos bx+\ln a\cdot\sin^2 bx+b\cdot\cos bx\cdot\sin bx-a^x\cdot\ln a\cdot\sin bx-b\cdot\sin bx\cdot\cos bx)=$.........2p

$= \frac{a^x}{(a^x+\sin bx)^2}\left(b \cdot a^x \cdot \cos bx + \ln a \cdot \sin^2 bx\right)$, de unde rezultă că

$I = \frac{a^x \cdot \sin bx}{a^x+\sin bx} + C$ ………………………………………………………………………………1p

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Total = 7 puncte

**3. a.** Facem schimbarea de variabilă $\sqrt{tg\ x} = t \;=>\; x = arctg\ t^2 \;=>\; dx = \frac{2t}{t^4+1}dt$

$I = \int t \cdot \frac{2t}{t^4+1}dt = \int \frac{2}{t^2+\frac{1}{t^2}}dt = \int \frac{1-\frac{1}{t^2}+1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = I_1 + I_2$

$$I_1 = \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}dt$$

Facem schimbarea de variabilă $t + \frac{1}{t} = u \;=>\; \left(1-\frac{1}{t^2}\right)dt = du$

$=> I_1 = \int \frac{du}{u^2-2} = \frac{1}{2\sqrt{2}} \cdot \ln\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|$ …………………………………………………………...……1p

$$I_2 = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$$

Facem schimbarea de variabilă $t - \frac{1}{t} = v \;=>\; \left(1+\frac{1}{t^2}\right)dt = dv$

$=> I_2 = \int \frac{dv}{v^2+2} = \frac{1}{\sqrt{2}} \cdot arctg\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)$ ……………………………………………………………..…1p

$=> I = \frac{1}{2\sqrt{2}} \cdot \ln\left(\frac{\sqrt{tg\ x}+\frac{1}{\sqrt{tg\ x}}-\sqrt{2}}{\sqrt{tg\ x}+\frac{1}{\sqrt{tg\ x}}+\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \cdot arctg\left(\frac{\sqrt{tg\ x}-\frac{1}{\sqrt{tg\ x}}}{\sqrt{2}}\right) + C$ ………………………….……….....1p

**b.** Facem schimbarea de variabilă $tg\ x = t \;=>\; \frac{1}{\cos^2 x}dx = dt$

$I = \int \frac{\sqrt{cos2x} \cdot \frac{1}{\cos^2 x}dx}{\sin x \cdot \frac{1}{\cos^2 x}} = \int \frac{\cos x \cdot \sqrt{cos2x}}{tg\ x} \cdot \frac{1}{\cos^2 x}dx$ ………………………………………...……....1p

$I = \int \frac{\frac{1}{\sqrt{t^2+1}} \cdot \frac{\sqrt{1-t^2}}{\sqrt{1+t^2}}dt}{t} = \int \frac{\sqrt{1-t^2}}{t(t^2+1)}dt$ ……………………………………………………………..……1p

Facem schimbarea de variabilă $\sqrt{1-t^2} = u =>$

$I = -\int \frac{u^2}{(u^2-2)(u^2-1)}du = \int \frac{1}{u^2-1}du - 2 \cdot \int \frac{1}{u^2-2}du =$ ……………………………...………..1p

$= \frac{1}{2} \cdot \ln\left|\frac{u-1}{u+1}\right| - \frac{2}{2\sqrt{2}} \cdot \ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| = \frac{1}{2} \cdot \ln\left|\frac{\sqrt{1-tg^2x}-1}{\sqrt{1-tg^2x}+1}\right| - \frac{\sqrt{2}}{2} \cdot \ln\left|\frac{\sqrt{1-tg^2x}-\sqrt{2}}{\sqrt{1-tg^2x}+\sqrt{2}}\right| + C$ ……....…..……1p

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Total = 7 puncte

**4. a.** $1 = (-1)^6 = -1, 1 = -1 \Rightarrow 1 + 1 = 0 \Rightarrow x(1 + 1) = x \cdot 0 = 0$
$\Rightarrow x + x = 0$..........................................................................................1p
$(x + 1)^6 = x + 1$
$C_6^0 x^6 + C_6^1 x^5 + C_6^2 x^4 + C_6^3 x^3 + C_6^4 x^2 + C_6^5 x + 1 = x + 1$
$x^6 + \underbrace{6x^5}_{=0} + \underbrace{15x^4}_{x^4} + \underbrace{20x^3}_{=0} + \underbrace{15x^2}_{x^2} + \underbrace{6x}_{=0} = x$
$x^6 + x^4 + x^2 = x$..........................................................................................1p
$x^4 + x^2 = 0 \quad | + x^2$
$x^4 + \underbrace{x^2 + x^2}_{=0} = x^2$
$x^4 = x^2 \ | \cdot x^2 \Leftrightarrow x^6 = x^4 = x^2 = x$
$\Leftrightarrow x = x^2 \ (\forall) x \in A$..........................................................................................1p

**b.** Pentru $x = a$ în egalitatea din enunţ, obţinem $a^4 = a$, deci $a^3 = e$ unde $e \in G$ este elementul neutru. Ipoteza devine: $x^4 = a^2xa$ ,pentru orice $x \in G$. (1).......................................1p
Fie $x \in G$. Din (1) rezultă că $xa = ax^4$ . Înlocuind $x$ cu $ax$ în (1), obţinem
$(ax)^4 = xa = ax^4$. Simplificând la stânga şi la dreapta egalitatea $(ax)^4 = ax^4$ .......................1p
deducem că $(xa)^3 = x^3$ . Rezultă $x(ax)^2a = x^3$ , deci $(ax)^2a = x^2$ şi $(ax)^3 = x^3$ ................1p
$(xa)^3 = (ax)^3$ . Înlocuind $x$ cu $a^2x$ în egalitatea precedentă, obţinem
$(a^2xa)^3 = x^3$ şi folosind (1) deducem că $(x^4)^3 = x^3$, deci $x^9 = e$ ..........................................1p

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Total = 7 puncte